

# The local equicontinuity of a maximal monotone operator

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## Abstract

The local equicontinuity of an operator  $T : X \rightrightarrows X^*$  with proper Fitzpatrick function  $\varphi_T$  and defined in a barreled locally convex space  $X$  has been shown to hold on the algebraic interior of  $\text{Pr}_X(\text{dom } \varphi_T)$ <sup>1</sup>. The current note presents direct consequences of the aforementioned result with regard to the local equicontinuity of a maximal monotone operator defined in a barreled locally convex space.

## 1 Introduction and notations

Throughout this paper, if not otherwise explicitly mentioned,  $(X, \tau)$  is a non-trivial (that is,  $X \neq \{0\}$ ) real Hausdorff separated locally convex space (LCS for short),  $X^*$  is its topological dual endowed with the weak-star topology  $w^*$ , the weak topology on  $X$  is denoted by  $w$ , and  $(X^*, w^*)^*$  is identified with  $X$ . The class of neighborhoods of  $x \in X$  in  $(X, \tau)$  is denoted by  $\mathcal{V}_\tau(x)$ .

The *duality product* or *coupling* of  $X \times X^*$  is denoted by  $\langle x, x^* \rangle := x^*(x) =: c(x, x^*)$ , for  $x \in X$ ,  $x^* \in X^*$ . With respect to the dual system  $(X, X^*)$ , the *polar* of  $A \subset X$  is  $A^\circ := \{x^* \in X^* \mid |\langle x, x^* \rangle| \leq 1, \forall x \in A\}$ .

A set  $B \subset X^*$  is  $(\tau-)$ *equicontinuous* if for every  $\epsilon > 0$  there is  $V_\epsilon \in \mathcal{V}_\tau(0)$  such that, for every  $x^* \in B$ ,  $x^*(V_\epsilon) \subset (-\epsilon, \epsilon)$ , or, equivalently,  $B$  is contained in the polar  $V^\circ$  of some (symmetric)  $V \in \mathcal{V}_\tau(0)$ .

A multi-function  $T : X \rightrightarrows X^*$  is  $(\tau-)$ locally equicontinuous<sup>2</sup> at  $x_0 \in X$  if there exists  $U \in \mathcal{V}_\tau(x_0)$  such that  $T(U) := \cup_{x \in U} Tx$  is a  $(\tau-)$ equicontinuous subset of  $X^*$ ;  $(\tau-)$ locally equicontinuous on  $S \subset X$  if  $T$  is  $(\tau-)$ locally equicontinuous at every  $x \in S$ . The  $(\tau-)$ local equicontinuity of  $T : X \rightrightarrows X^*$  is interesting only at  $x_0 \in \overline{D(T)}^\tau$  ( $\tau$ -closure) since, for every  $x_0 \notin \overline{D(T)}^\tau$ ,  $T(U)$  is void for a certain  $U \in \mathcal{V}_\tau(x_0)$ . Here  $\text{Graph } T = \{(x, x^*) \in X \times X^* \mid x^* \in Tx\}$  is the graph of  $T$ ,  $D(T) = \text{Pr}_X(\text{Graph } T)$  stands for the domain of  $T$ , where  $\text{Pr}_X$  denotes the projection of  $X \times X^*$  onto  $X$ .

<sup>1</sup>see [5, Theorem 4]

<sup>2</sup>“locally bounded” is the notion used in the previous papers on the subject (see e.g. [5, Definition 1], [3, p. 397]), mainly because in the context of barreled spaces, weak-star bounded and equicontinuous subsets of the dual coincide.

The main objective of this paper is to give a description of the local equicontinuity set (see Definition 1 below) of a maximal monotone operator  $T : X \rightrightarrows X^*$  ( $T \in \mathfrak{M}(X)$  for short) defined in a barreled space  $X$  in terms of its *Fitzpatrick function*  $\varphi_T : X \times X^* \rightarrow \overline{\mathbb{R}}$  which is given by (see [1])

$$\varphi_T(x, x^*) := \sup\{\langle x - a, a^* \rangle + \langle a, x^* \rangle \mid (a, a^*) \in \text{Graph } T\}, \quad (x, x^*) \in X \times X^*. \quad (1)$$

As usual, given a LCS  $(E, \mu)$  and  $A \subset E$  we denote by “conv  $A$ ” the *convex hull* of  $A$ , “ $\text{cl}_\mu(A) = \overline{A}^\mu$ ” the  $\mu$ -closure of  $A$ , “ $\text{int}_\mu A$ ” the  $\mu$ -topological interior of  $A$ , “core  $A$ ” the *algebraic interior* of  $A$ . The use of the  $\mu$ -notation is enforced barring that the topology  $\mu$  is clearly understood.

When  $X$  is a Banach space, Rockafellar showed in [3, Theorem 1] that if  $T \in \mathfrak{M}(X)$  has  $\text{int}(\text{conv } D(T)) \neq \emptyset$  then  $\overline{D(T)}$  is *nearly-solid* in the sense that  $\text{int } D(T)$  is non-empty, convex, whose closure is  $\overline{D(T)}$ , while  $T$  is locally equicontinuous (equivalently said, locally bounded) at each point of  $\text{int } D(T)$  and unbounded at any boundary point of  $D(T)$ .

Our results extend the results of Rockafellar [3] to the framework of barreled LCS's or present new shorter arguments for known ones (see Theorems 2, 5, 8, 9 below).

## 2 The local equicontinuity set

One of the main reasons for the usefulness of equicontinuity is that it ensures the existence of a limit for the duality product on the graph of an operator. More precisely, if  $T : X \rightrightarrows X^*$  is locally equicontinuous at  $x_0 \in \overline{D(T)}^\tau$  then for every net  $\{(x_i, x_i^*)\}_i \subset \text{Graph } T$  with  $x_i \xrightarrow{\tau} x_0$ ,  $\{x_i^*\}_i$  is equicontinuous thus, according to Bourbaki's Theorem, weak-star relatively compact in  $X^*$  and, at least on a subnet,  $x_i^* \rightarrow x_0^*$  weak-star in  $X^*$  and  $\lim_i c(x_i, x_i^*) = c(x_0, x_0^*)$ .

**Definition 1** Given  $(X, \tau)$  a LCS, for every  $T : X \rightrightarrows X^*$  we denote by

$$\Omega_T^{(\tau)} := \{x \in \overline{D(T)}^\tau \mid T \text{ is } (\tau-)\text{locally equicontinuous at } x\}$$

the (*meaningful*)  $(\tau-)$ local equicontinuity set of  $T$ .

In the above notation, our local equicontinuity result [5, Theorem 8] states that if the LCS  $(X, \tau)$  is barreled then

$$\forall T : X \rightrightarrows X^*, \quad \text{core } \text{Pr}_X(\text{dom } \varphi_T) \cap \overline{D(T)}^\tau \subset \Omega_T^{(\tau)}. \quad (2)$$

**Theorem 2** If  $(X, \tau)$  is a barreled LCS and  $T \in \mathfrak{M}(X)$  has  $\overline{D(T)}^\tau$  convex then

$$\text{int}_\tau \text{Pr}_X(\text{dom } \varphi_T) = \text{core } \text{Pr}_X(\text{dom } \varphi_T). \quad (3)$$

*The following are equivalent*

(i)  $\text{core Pr}_X(\text{dom } \varphi_T) \neq \emptyset$ , (ii)  $\Omega_T^\tau \neq \emptyset$  and  $\text{core } \overline{D(T)}^\tau \neq \emptyset$ , (iii)  $\text{int}_\tau D(T) \neq \emptyset$ .

In this case

$$\Omega_T^\tau = \text{core Pr}_X(\text{dom } \varphi_T) = \text{int}_\tau D(T) \quad (4)$$

and  $D(T)$  is  $\tau$ -nearly-solid in the sense that  $\text{int}_\tau D(T) = \text{int}_\tau \overline{D(T)}^\tau$  and  $\overline{D(T)}^\tau = \text{cl}_\tau(\text{int}_\tau D(T))$ .

In the proof of the Theorem 2 we use the following lemma.

**Lemma 3** *Let  $(X, \tau)$  be a LCS and  $T : X \rightrightarrows X^*$ . Then*

$$\text{Pr}_X(\text{dom } \varphi_T) \subset D(T^+) \cup \overline{\text{conv}}^\tau(D(T)). \quad (5)$$

Here  $T^+ : X \rightrightarrows X^*$  is the operator associated to the set of elements monotonically related to  $T$ ,  $\text{Graph } T^+ = \{(x, x^*) \in X \times X^* \mid \langle x - a, x^* - a^* \rangle \geq 0, \forall (a, a^*) \in \text{Graph } T\}$ .

If, in addition,  $T \in \mathcal{M}(X)$ , then

$$\text{Pr}_X(\text{dom } \varphi_T) \subset D(T^+) \cup \overline{\text{conv}}^\tau(D(T)) \subset \text{cl}_\tau \text{Pr}_X(\text{dom } \varphi_T). \quad (6)$$

If, in addition,  $T \in \mathfrak{M}(X)$ , then

$$\text{cl}_\tau \text{Pr}_X(\text{dom } \varphi_T) = \overline{\text{conv}}^\tau(D(T)). \quad (7)$$

If, in addition,  $T \in \mathfrak{M}(X)$  and  $\overline{D(T)}^\tau$  is convex then

$$\text{cl}_\tau \text{Pr}_X(\text{dom } \varphi_T) = \overline{D(T)}^\tau. \quad (8)$$

**Proof.** Let  $T : X \rightrightarrows X^*$  and  $(x, x^*) \in \text{dom } \varphi_T$  with  $x \notin \overline{\text{conv}}^\tau(D(T))$ . Separate to get  $y^* \in X^*$  such that  $\inf\{\langle x - a, y^* \rangle \mid a \in D(T)\} \geq (\varphi_T - c)(x, x^*)$ . For every  $(a, a^*) \in T$  we have

$$\langle x - a, x^* + y^* - a^* \rangle \geq (c - \varphi_T)(x, x^*) + \langle x - a, y^* \rangle \geq 0,$$

i.e.,  $(x, x^* + y^*) \in \text{Graph } T^+$  so  $x \in D(T^+)$ . Therefore the first inclusion in (6) holds for every  $T : X \rightrightarrows X^*$  while the second inclusion in (6) follows from  $D(T) \subset \text{Pr}_X(\text{dom } \varphi_T)$  and  $\text{Graph } T^+ \subset \text{dom } \varphi_T$ .

If, in addition,  $T \in \mathfrak{M}(X)$ , then  $T = T^+$  and (6) translates into  $\text{cl}_\tau \text{Pr}_X(\text{dom } \varphi_T) = \overline{\text{conv}}^\tau(D(T))$ . ■

**Proof of Theorem 2.** According to Lemma 3,  $\text{cl}_\tau(\text{Pr}_X(\text{dom } \varphi_T)) = \overline{D(T)}^\tau$ .

Because  $T \in \mathfrak{M}(X)$  we claim that

$$\forall x \in \Omega_T^\tau, \exists U \in \mathcal{U}_\tau(x), T(U) \text{ is equicontinuous and } U \cap \overline{D(T)}^\tau \subset D(T). \quad (9)$$

In particular  $\Omega_T^\tau \subset D(T)$ .

Indeed, for every  $x \in \Omega_T^\tau$  take  $U$  a  $\tau$ -open neighborhood of  $x$  such that  $T(U)$  is  $(\tau)$ -equicontinuous; in particular  $T(U)$  is weak-star relatively compact. For every  $y \in$

$U \cap \overline{D(T)}^\tau$  there is a net  $(y_i)_{i \in I} \subset U \cap D(T)$  such that  $y_i \xrightarrow{\tau} y$ . Take  $y_i^* \in Ty_i, i \in I$ . At least on a subnet  $y_i^* \rightarrow y^*$  weakly-star in  $X^*$ . For every  $(a, a^*) \in T, \langle y_i - a, y_i^* - a^* \rangle \geq 0$ , because  $T \in \mathcal{M}(X)$ . After we pass to limit, taking into account that  $(y_i^*)$  is  $(\tau-)$ equicontinuous, we get  $\langle y - a, y^* - a^* \rangle \geq 0$ , for every  $(a, a^*) \in T$ . This yields  $(y, y^*) \in T$  due to the maximality of  $T$ ; in particular  $y \in D(T)$ . Therefore  $U \cap \overline{D(T)}^\tau \subset D(T)$ .

Whenever  $\text{core } \overline{D(T)}^\tau$  is non-empty,  $\text{int}_\tau \overline{D(T)}^\tau = \text{core } \overline{D(T)}^\tau \neq \emptyset$ , since  $(X, \tau)$  is barreled. This yields that  $\overline{D(T)}^\tau$  is solid, so,  $\overline{D(T)}^\tau = \text{cl}_\tau(\text{int}_\tau \overline{D(T)}^\tau)$  (see e.g. [2, Lemma p. 59]).

Whenever  $\Omega_T^\tau \neq \emptyset$  and  $\text{core } \overline{D(T)}^\tau \neq \emptyset$ , from (9),  $D(T)$  contains the non-empty open set  $U \cap \text{int}_\tau \overline{D(T)}^\tau$ ;  $\text{int}_\tau D(T) \neq \emptyset$  and so  $\text{int}_\tau \text{Pr}_X(\text{dom } \varphi_T) \neq \emptyset$  which also validates (3).

From the above considerations and

$$\text{int}_\tau D(T) \subset \text{core } \text{Pr}_X(\text{dom } \varphi_T) \subset \Omega_T^\tau \cap \text{core } \overline{D(T)}^\tau$$

we see that (iii)  $\Rightarrow$  (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii).

In this case, since  $\text{Pr}_X(\text{dom } \varphi_T)$  is convex, according to [2, Lemma p. 59],

$$\text{core } \text{Pr}_X(\text{dom } \varphi_T) = \text{int}_\tau \text{Pr}_X(\text{dom } \varphi_T) = \text{int}_\tau(\text{cl}_\tau(\text{Pr}_X(\text{dom } \varphi_T))) = \text{int}_\tau \overline{D(T)}^\tau.$$

Hence

$$\text{int}_\tau D(T) \subset \text{core } D(T) \subset \text{core } \text{Pr}_X(\text{dom } \varphi_T) = \text{int}_\tau \text{Pr}_X(\text{dom } \varphi_T) \subset \Omega_T^\tau \subset D(T)$$

followed by  $\text{int}_\tau D(T) = \text{core } D(T) = \text{core } \text{Pr}_X(\text{dom } \varphi_T) = \text{int}_\tau \overline{D(T)}^\tau$ .

It remains to prove that  $\Omega_T^\tau \subset \text{int}_\tau D(T)$ . Assume, by contradiction that  $x \in \Omega_T^\tau \setminus \text{int}_\tau D(T)$ . Since, in this case,  $x$  is a support point for the convex set  $\overline{D(T)}^\tau$  whose interior is non-empty, we know that  $N_{\overline{D(T)}^\tau}\{x\} \neq \{0\}$ . Together with  $T = T + N_{\overline{D(T)}^\tau}$ , this yields that  $Tx$  is unbounded; in contradiction to the fact that  $Tx$  is equicontinuous. ■

Recall that a set  $S \subset X$  is *nearly-solid* if there is a convex set  $C$  such that  $\text{int } C \neq \emptyset$  and  $C \subset S \subset \overline{C}$ . Equivalently,  $S$  is nearly-solid iff  $\text{int } S$  is non-empty convex and  $S \subset \text{cl}(\text{int } S)$ . Indeed, directly, from  $\text{int } C = \text{int } \overline{C}$  and  $\text{cl}(\text{int } C) = \overline{C}$  (see [2, Lemma 11A b), p. 59]) we know that  $\text{int } S = \text{int } C$  is non-empty convex and  $S \subset \overline{C} = \text{cl}(\text{int } S)$ . Conversely,  $C = \text{int } S$  fulfills all the required conditions.

**Corollary 4** *If  $(X, \tau)$  is a barreled LCS and  $T \in \mathfrak{M}(X)$  has  $\text{int}_\tau D(T) \neq \emptyset$  and  $\overline{D(T)}^\tau$  convex then  $D(T)$  is nearly-solid and  $\Omega_T^\tau = \text{int}_\tau D(T)$ .*

**Theorem 5** *If  $(X, \tau)$  is a barreled normed space and  $T \in \mathfrak{M}(X)$  has  $\text{core } \text{Pr}_X(\text{dom } \varphi_T) \neq \emptyset$  then  $D(T)$  is nearly-solid and  $\Omega_T^\tau = \text{core } \text{Pr}_X(\text{dom } \varphi_T) = \text{int}_\tau D(T)$ .*

**Proof.** It suffices to prove that  $\overline{D(T)}^\tau$  is convex. We actually prove that

$$\text{core } \text{Pr}_X(\text{dom } \varphi_T) \subset D(T);$$

from which  $\overline{D(T)}^\tau = \text{cl}_\tau(\text{core Pr}_X(\text{dom } \varphi_T)) = \text{cl}_\tau \text{Pr}_X(\text{dom } \varphi_T)$  is convex.

For a fixed  $z \in \text{core Pr}_X(\text{dom } \varphi_T)$  consider the function  $\Phi : X^* \times X \rightarrow \overline{\mathbb{R}}$ ,  $\Phi(x^*, x) = \varphi_T(x + z, x^*) - \langle z, x^* \rangle$ . Then  $0 \in \text{core}(\text{Pr}_X(\text{dom } \Phi))$ . The function  $\Phi$  is convex, proper, and  $w^* \times \tau$ -lsc so it is also  $s^* \times \tau$ -lsc, where  $s^*$  denotes the strong topology on  $X^*$ .

We may use [7, Proposition 2.7.1 (vi), p. 114] to get

$$\inf_{x^* \in X^*} \Phi(x^*, 0) = \max_{y^* \in X^*} (-\Phi^*(0, y^*)). \quad (10)$$

Because  $\varphi_T \geq c$  (see e.g. [1, Corollary 3.9]),  $\inf_{x^* \in X^*} \Phi(x^*, 0) \geq 0$  and that  $\Phi^*(x^{**}, x^*) = \varphi_T^*(x^*, x^{**} + z) - \langle z, x^* \rangle$ ,  $x^* \in X^*$ ,  $x^{**} \in X^{**}$ . Therefore, from (10), there exists  $y^* \in X^*$  such that  $\varphi_T^*(y^*, z) =: \psi_T(z, y^*) \leq \langle z, y^* \rangle$ , that is,  $(z, y^*) \in [\psi_T \leq c] = [\psi_T = c] = T$  (see [4, Theorem 2.2] or [6, Theorem 1] for more details). Hence  $z \in D(T)$ . ■

**Corollary 6** *If  $X$  is a barreled normed space and  $T \in \mathfrak{M}(X)$  has  $\text{core}(\text{conv } D(T)) \neq \emptyset$  then  $D(T)$  is nearly-solid and  $\Omega_T = \text{int } D(T)$ .*

**Corollary 7** *If  $X$  is a Banach space and  $T \in \mathfrak{M}(X)$  has  $\text{int}(\text{conv } D(T)) \neq \emptyset$  then  $D(T)$  is nearly-solid and  $\Omega_T = \text{int } D(T)$ .*

Under a Banach space settings the proof of the following result is known from [3, p. 406] (see also [7, Theorem 3.11.15, p. 286]) and it relies on the Bishop-Phelps Theorem, namely, on the density of the set of support points to a closed convex set in its boundary. The novelty of our argument is given by the use of the maximal monotonicity of the normal cone to a closed convex set.

**Theorem 8** *Let  $X$  be a Banach space and let  $T \in \mathfrak{M}(X)$  be such that  $\overline{D(T)}$  is convex and  $\Omega_T \neq \emptyset$ . Then  $\Omega_T = \text{int } D(T)$ .*

**Proof.** For every  $x \in \Omega_T$ , let  $U \in \mathcal{V}(x)$  be closed convex as in (9). Since  $T + N_{\overline{D(T)}} = T$  and  $T(U)$  is equicontinuous we know that there are no support points to  $\overline{D(T)}$  in  $U$ , i.e., for every  $u \in \overline{D(T)} \cap U = D(T) \cap U$ ,  $N_{\overline{D(T)}}|_U(u) = \{0\}$ .

Then

$$N_{\overline{D(T)} \cap U} = N_{\overline{D(T)}} + N_U = N_U|_{D(T)} \subset N_U,$$

and since  $N_{\overline{D(T)} \cap U} \in \mathfrak{M}(X)$ ,  $N_U \in \mathcal{M}(X)$  we get that  $U \subset D(T)$ , and so  $x \in \text{int } D(T)$ . ■

We conclude this paper with two results on the convex subdifferential.

**Theorem 9** *Let  $(X, \tau)$  be a LCS and let  $f : X \rightarrow \overline{\mathbb{R}}$  be proper convex  $\tau$ -lower semicontinuous such that  $f$  is continuous at some  $x_0 \in \text{int}_\tau(\text{dom } f)$ . Then  $\partial f \in \mathfrak{M}(X)$ ,  $D(\partial f)$ ,  $\text{dom } f$  are nearly-solid,  $\text{int}_\tau(\text{dom } f) = \text{int}_\tau D(\partial f)$ ,  $\text{cl}_\tau(\text{dom } f) = \overline{D(\partial f)}^\tau$ , and  $\Omega_{\partial f}^\tau = \text{int}_\tau D(\partial f)$ .*

**Proof.** Recall that in this case  $f$  is continuous on  $\text{int}_\tau(\text{dom } f)$ ,  $\text{dom } f$  is a solid set, and  $\text{int}_\tau(\text{dom } f) \subset D(\partial f) \subset \text{dom } f$  (see e.g. [7, Theorems 2.2.9, 2.4.12]). The latter inclusions implies that  $D(\partial f)$  is nearly-solid, and  $\text{int}_\tau(\text{dom } f) = \text{int}_\tau D(\partial f)$ ,  $\text{cl}_\tau(\text{dom } f) = \overline{D(\partial f)}^\tau$ .

Clearly,  $f(x) + f^*(x^*)$  is a representative of  $\partial f$  so  $\partial f$  is representable. In order for  $\partial f \in \mathfrak{M}(X)$  it suffices to prove that  $\partial f$  is of negative infimum type (see [6, Theorem 1(ii)] or [4, Theorem 2.3]). Seeking a contradiction, assume that  $(x, x^*) \in X \times X^*$  satisfies  $\varphi_{\partial f}(x, x^*) < \langle x, x^* \rangle$ ; in particular  $(x, x^*)$  is monotonically related to  $\partial f$ , i.e., for every  $(u, u^*) \in \partial f$ ,  $\langle x - u, x^* - u^* \rangle \geq 0$ . Let  $g : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ ,  $g(t) := f(tx + (1 - t)x_0) = f(Lt + x_0)$ , where  $L : \mathbb{R} \rightarrow X$ ,  $Lt = t(x - x_0)$ . According to the chain rule

$$\partial g(t) = L^*(\partial f(Lt + x_0)), \text{ or } s \in \partial g(t) \Leftrightarrow \exists y^* \in \partial f(tx + (1 - t)x_0), \langle x - x_0, y^* \rangle = s.$$

The function  $g$  is proper convex lower semicontinuous so  $\partial g \in \mathfrak{M}(\mathbb{R})$ . But,  $(1, \langle x - x_0, x^* \rangle)$  is monotonically related to  $\partial g$  because, for every  $s \in \partial g(t)$  there is  $y^* \in \partial f(tx + (1 - t)x_0)$  such that  $\langle x - x_0, y^* \rangle = s$  which provides  $(1 - t)(\langle x, x^* \rangle - s) = \langle x - (tx + (1 - t)x_0), x^* - y^* \rangle \geq 0$ . Therefore  $(1, \langle x - x_0, x^* \rangle) \in \partial g$ , in particular  $1 \in D(\partial g)$  and  $x \in D(\partial f)$ . That implies the contradiction  $\varphi_{\partial f}(x, x^*) \geq \langle x, x^* \rangle$ .

Hence  $\varphi_{\partial f}(x, x^*) \geq \langle x, x^* \rangle$ , for every  $(x, x^*) \in X \times X^*$ , that is,  $\partial f$  is of negative infimum type and consequently maximal monotone.

Also  $\partial f$  is locally equicontinuous on  $\text{int}_\tau(\text{dom } f)$  which shows that  $\text{int}_\tau D(\partial f) \subset \Omega_{\partial f}^\tau$  (see e.g. [7, Theorem 2.2.11] and the proof of [7, Theorem 2.4.9]).

Because  $\partial f \in \mathfrak{M}(X)$ , we see, as in the proof of Theorem 2, that  $\Omega_{\partial f}^\tau \subset D(\partial f)$ . From  $f = f + \iota_{\text{dom } f}$  we get  $\partial f = \partial f + N_{\overline{D(\partial f)}}$  which shows that  $\partial f(x)$  is unbounded for each  $x \in D(\partial f) \setminus \text{int}_\tau D(\partial f)$  since  $N_{\overline{D(\partial f)}}(x) \neq \{0\}$ . Hence  $\Omega_{\partial f}^\tau = \text{int}_\tau D(\partial f)$ . ■

**Theorem 10** *Let  $(X, \tau)$  be a barreled LCS and let  $f : X \rightarrow \overline{\mathbb{R}}$  be proper convex  $\tau$ -lower semicontinuous with  $\text{core}(\text{dom } f) \neq \emptyset$ . Then  $\partial f \in \mathfrak{M}(X)$ ,  $D(\partial f)$ ,  $\text{dom } f$  are nearly-solid,  $\text{int}_\tau(\text{dom } f) = \text{int}_\tau D(\partial f)$ ,  $\text{cl}_\tau(\text{dom } f) = \overline{D(\partial f)}^\tau$ , and  $\Omega_{\partial f}^\tau = \text{int}_\tau D(\partial f)$ .*

**Proof.** It suffices to notice that under a barreled space context  $\text{core}(\text{dom } f) = \text{int}_\tau(\text{dom } f)$ ,  $f$  is continuous on  $\text{int}_\tau(\text{dom } f)$  (see e.g. [7, Theorem 2.2.20]), and we may apply Theorem 9. ■

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